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On the best constants for the Brezis–Marcus inequalities in balls[☆]

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ABSTRACT

We study the best possible constants $c(n)$ in the Brezis–Marcus inequalities

$$\int_{B_n} |\nabla u|^2\, dx \ge \frac{1}{4}\int_{B_n} \frac{|u|^2}{(\rho - |x-x_0|)^2}\, dx + \frac{c(n)}{\rho^2}\int_{B_n} |u|^2\, dx$$

for $u \in H_0^1(B_n)$ in balls $B_n = \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$. The quantity $c(1)$ is known by our paper [F.G. Avkhadiev, K.-J. Wirths, Unified Poincaré and Hardy inequalities with sharp constants for convex domains, ZAMM Z. Angew. Math. Mech. 87 (8–9) 26 (2007) 632–642]. In the present paper we prove the estimate $c(2) \ge 2$ and the assertion

$$\lim_{n\to\infty} \frac{c(n)}{n^2} = \frac{1}{4},$$

which gives that the known lower estimates in [G. Barbatis, S. Filippas, and A. Tertikas in Comm. Cont. Math. **5** (2003), no. 6, 869–881] for $c(n)$, $n \ge 3$, are asymptotically sharp as $n \to \infty$. Also, for the 3-dimensional ball $B_3^0 = \{x \in \mathbb{R}^3 : |x| < 1\}$ we obtain a new Brezis–Marcus type inequality which contains two parameters $m \in (0, \infty)$, $\nu \in (0, 1/m)$ and has the following form

$$\int_{B_3^0} |\nabla u(x)|^2\, dx \ge \frac{1}{4}\int_{B_3^0} \left\{ \frac{1-\nu^2 m^2}{(1-|x|)^2} + \frac{m^2 j_\nu^2}{(1-|x|)^{2-m}} \right\} |u(x)|^2\, dx,$$

where j_ν is the first zero of the Bessel function J_ν of order ν and the constants

$$\frac{1-\nu^2 m^2}{4} \quad \text{and} \quad \frac{m^2 j_\nu^2}{4}$$

are sharp for all admissible values of parameters m and ν.

1. Introduction

Consider first the Hardy inequality for n-dimensional convex domains Ω

$$\int_\Omega |\nabla u|^2\, dx \ge \frac{1}{4}\int_\Omega \frac{|u|^2}{\delta^2}\, dx, \quad \forall u \in H_0^1(\Omega),$$

where $\delta = dist(x, \partial\Omega)$, and the space $H_0^1(\Omega)$ is the closure of the family $C_0^1(\Omega)$ of smooth functions $u : \Omega \to \mathbb{R}$ with finite Dirichlet integral and supported in Ω. Although there is no function $u \not\equiv 0$, $u \in H_0^1(\Omega)$, for which the equality in the

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